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Mathematical Methods Examination 2

Solutions Book

VCE Trial Examination – Free

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Section A – Multiple-choice questions

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Question 1 (B)

The range of f is $[-3-2, -3+2] = [-5, -1]$.

Question 2 (C)

The distance between the points is

$$d = \sqrt{\big(1-(-2)\big)^2 + (3-6)^2} = 3\sqrt{2}.$$

Question 3 (D)

The gradient of the line is $-\frac{1}{\sqrt{3}}$ and so the angle $\theta \in (0, \pi)$ is given by

$$\tan(\theta) = -\frac{1}{\sqrt{3}} \implies \theta = \frac{5\pi}{6}.$$

Question 4 (A)

The probability that the marbles are the same colour is

$$\Pr(S) = \Pr\left(\{(R,R)\}\right) + \Pr\left(\{(B,B)\}\right) = \frac{4}{10}\cdot\frac{3}{9} + \frac{6}{10}\cdot\frac{5}{9} = \frac{7}{15}.$$

Question 5 (A)

Observe that

$$g(x) = \left(\frac{x-1}{2}\right)^2 - \frac{x-1}{2} - 1 = \frac{x^2}{4} - x - \frac{1}{4},$$

and so the rule of $f \circ g$ is given by

$$(f \circ g)(x) = f\big(g(x)\big) = 1 - 4\left(\frac{x^2}{4} - x - \frac{1}{4}\right) = -x^2 + 4x + 2.$$

Question 6 (C)

The system has a non-unique solution when

$$m(m+1) - 3\cdot 4 = 0 \implies m = -4 \text{ or } m = 3.$$

If $m = -4$, then the equations become $x - y = -\frac{10}{3}$ and $x - y = 1$, which has no solutions. If $m = 3$, then the equations become $4x + 3y = 10$ and $4x + 3y = 4$, which also has no solutions. Therefore, the system has no solutions if and only if $m \in \{-4, 3\}$.

Question 7 (A)

The graph of f lies strictly above the graph of g and so $f-g$ must be positive for this region.

Question 8 (B)

The variance of X is given by

$$\begin{aligned}\mathrm{Var}(X) &= (-1)^2\cdot 0.2+0^2\cdot 0.25+1^2\cdot 0.4+2^2\cdot 0.15-(-1\cdot 0.2+0\cdot 0.25+1\cdot 0.4+2\cdot 0.15)^2\\ &= 0.95.\end{aligned}$$

Question 9 (C)

Using known integral properties,

$$\begin{aligned}\int_2^4 \big(3f(x)-x\big)\,dx &= 3\int_2^4 f(x)\,dx-\int_2^4 x\,dx\\ &= 3\left(\int_0^4 f(x)\,dx-\int_0^2 f(x)\,dx\right)-\int_2^4 x\,dx\\ &= 3\cdot(10-4)-6\\ &= 12.\end{aligned}$$

Question 10 (B)

Since A and B are independent, so too are A' and B'. Therefore,

$$\Pr(A'\cap B')=\Pr(A')\Pr(B')=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}.$$

Question 11 (A)

We have that for all $x\in\mathbb{R}$,

$$2\,(f(x))^2-1=2\left(\frac{e^x+e^{-x}}{2}\right)^2-1=2\cdot\frac{e^{2x}+e^{-2x}+2}{4}-1=\frac{e^{2x}+e^{-2x}}{2}=f(2x).$$

Question 12 (C)

The points of inflection in this case are located where the second derivative vanishes. That is, where

$$\frac{d^2y}{dx^2}=-\cos\left(\frac{x}{2}\right)=0 \implies x=(2k-1)\pi,\ \ k\in\mathbb{Z}.$$

Question 13 (D)

By the fundamental theorem of calculus,

$$\int_2^5 f'(x)\,dx=f(5)-f(2) \implies f(5)=\int_2^5 f'(x)\,dx-3.$$

Question 14 (C)

We have

$$\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1) = 1 - \frac{3}{e^2} \approx 0.5940 \quad \text{(4DP)}.$$

Question 15 (A)

We have $p'(x) = 3x^2 + 2bx + c$, and so the graph of p has two stationary points when

$$\Delta = (2b)^2 - 4 \cdot 3c > 0 \implies c < \frac{b^2}{3}.$$

Question 16 (B)

This program performs Newton's method with a tolerance of 0.001. Observe that

$$x_1 = -0.75 \implies x_2 = -0.6860... \implies x_3 = -0.6823... \implies x_4 = -0.6823...$$

The number of times the **`while`** loop will run is equal to the number of iterations after which the first three decimal places of the estimation does not change. Therefore, it loops four times.

Question 17 (D)

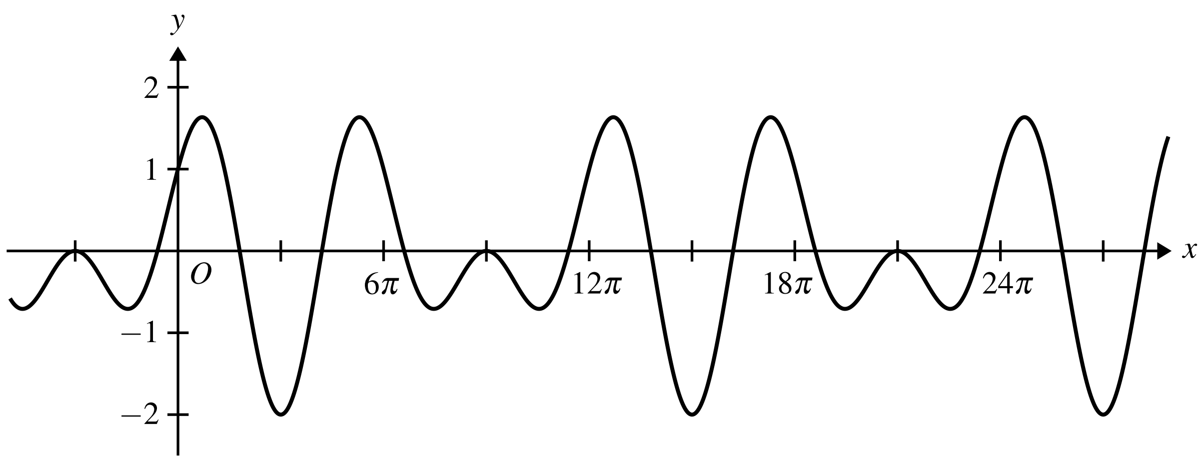
The area of the shaded region is given by

$$\int_{-a}^{a} \big(f(x) - f(a)\big)\,dx = \int_{-a}^{a} f(x)\,dx - f(a)\int_{-a}^{a} dx = 2\int_{0}^{a} f(x)\,dx - 2af(a),$$

where in the final equality, we used the fact that f is even.

Question 18 (D)

From the graph of h, the period is 12π.



Alternatively: The period of $h_1(x) = \sin\left(\frac{x}{2}\right)$ is 4π and the period of $h_2(x) = \cos\left(\frac{x}{3}\right)$ is 6π. The period of h is therefore $\text{lcm}(4,6)\pi = 12\pi$.

Question 19 (B)

Rearranging the inequality gives

$$\begin{aligned}\Pr(X \geq 1) = 1 - \binom{n}{0} p^0 (1-p)^{n-0} > 0.9 &\implies (1-p)^n < 0.1 \\ &\implies n\log_e(1-p) < \log_e(0.1) \\ &\implies n > \frac{\log_e(0.1)}{\log_e(1-p)},\end{aligned}$$

noting that $\log_e(1-p) < 0$ for all $p \in (0,1)$.

Question 20 (A)

By the inverse function theorem,

$$(f^{-1})'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(-1)} = -\frac{1}{2}.$$

End of Section A

Section B

Question 1a (1 mark)

MARK 1. Provides correct answer

The derivative of f is

$$f'(x) = 4x^3 - 16x.$$

Question 1b.i (2 marks)

MARK 1. Finds x-values where f is minimum
MARK 2. Finds minimum value of f

Solving $f'(x) = 0$ gives $x = -2,\ 0,\ 2$. Using the graph, the minima occur where $x = \pm 2$ and $f_{\text{min}} = -16$.

Question 1b.ii (1 mark)

MARK 1. Provides correct answer

The graph of f has a local maximum at $(0, 0)$.

Question 1b.iii (1 mark)

MARK 1. Provides correct answer

The parameter p acts as a vertical translation on the graph of f. The equation has exactly two solutions if and only if $f(0) + p < 0$ or $f(2) + p = 0$. That is, $p \in (-\infty, 0) \cup \{16\}$.

Question 1c (2 marks)

MARK 1. Expresses total area as integral
MARK 2. Provides correct answer

The total area of the regions is given by

$$A = -\int_{-2\sqrt{2}}^{2\sqrt{2}} f(x)\, dx = \frac{512\sqrt{2}}{15}.$$

Question 1d (1 mark)

MARK 1. Provides correct answer

The functions equal when $a = 0$.

Question 1e (1 mark)

MARK 1. Provides correct answers

Solving $g'(x) = 0$ gives $x = -\frac{a}{8},\ \pm 2$.

Question 1f.i (1 mark)

MARK 1. Provides correct answer

The minimum number of stationary points the graph of g can have is two.

Question 1f.ii (2 marks)

MARK 1. Identifies how graph can have a stationary point of inflection
MARK 2. Provides correct answer

The graph of g will have a stationary point of inflection whenever g' has a double root. That is when

$$-\frac{a}{8} = \pm 2 \implies a = \pm 16.$$

Question 1f.iii (1 mark)

MARK 1. Provides correct answer

The minimum of g occurs where $x = 2$ when $a = 16$, and where $x = -2$ when $a = -16$. In either case,

$$\operatorname{ran}(g) = \left[-\frac{176}{3}, \infty\right).$$

Question 2a (1 mark)

MARK 1. Provides correct answer

We have $C(20) = 2.53$ mg L^{-1} (2DP).

Question 2b (1 mark)

MARK 1. Provides correct answer

The derivative of C is given by

$$C'(t) = \frac{1}{75}(25 - t)e^{-\frac{t}{15}}.$$

Question 2c (1 mark)

MARK 1. Provides correct answer

Since $C'(25) = 0$, using the graph, the largest interval on which C is strictly increasing is $[0, 25]$.

Question 2d (1 mark)

MARK 1. Provides correct answer

The maximum concentration is $C_{\max} = C(25) = 2.57$ mg L^{-1} (2DP).

Question 2e (1 mark)

MARK 1. Provides correct answer

The limiting concentration is $\lim_{t\to\infty} C(t) = 2$ mg L^{-1}.

Question 2f (2 marks)

MARK 1. Expresses average rate of change as difference quotient, or equivalent merit
MARK 2. Provides correct answer

The average rate of change over [0, 25] is

$$\overline{C}' = \frac{C(25) - C(0)}{25 - 0} = 0.103 \text{ mg L}^{-1} \text{ min}^{-1} \quad \text{(3DP)}.$$

Question 2g (2 marks)

MARK 1. Expresses average concentration as integral
MARK 2. Provides correct answer

The average concentration over [0, 25] is

$$\overline{C} = \frac{1}{25 - 0} \int_0^{25} C(t)\,dt = 1.92 \text{ mg L}^{-1} \quad \text{(2DP)}.$$

Question 2h (2 marks)

MARK 1. Finds $C''(t)$, or equivalent merit
MARK 2. Provides correct answer

The minimum value of C' occurs when

$$C''(t) = \frac{1}{1125}(t - 40)e^{-\frac{t}{15}} = 0 \implies t = 40 \text{ min}.$$

Question 3a (1 mark)

MARK 1. Provides correct answer

The period of f is 6π.

Question 3b (2 marks)

MARK 1. Reasons valid method
MARK 2. Provides correct answer

Observe that for $x \in [0, 12\pi]$, we have

$$f(x) = 2 \implies x = \frac{3\pi}{2}, \frac{9\pi}{2}, \frac{15\pi}{2}, \frac{21\pi}{2},$$

so using the graph, $f(x) < 2$ on

$$\left(\frac{3\pi}{2}, \frac{9\pi}{2}\right) \cup \left(\frac{15\pi}{2}, \frac{21\pi}{2}\right).$$

Question 3c (2 marks)

MARK 1. Applies trapezium rule
MARK 2. Provides correct answer

By the trapezium rule,

$$A \approx \frac{3\pi - 0}{2 \cdot 6}\left[f(0) + 2f\left(\frac{\pi}{2}\right) + 2f(\pi) + 2f\left(\frac{3\pi}{2}\right) + 2f(2\pi) + 2f\left(\frac{5\pi}{2}\right) + f(3\pi)\right] = \frac{15\pi}{2}.$$

Question 3d (2 marks)

MARK 1. Expresses rule of g in terms of rule of f, or equivalent merit
MARK 2. Shows sufficient and correct algebraic work to arrive at conclusion

From the sequence of transformations and complementary angle, we have

$$\begin{aligned} g(x) = 10 - f\left(x - \frac{3\pi}{2}\right) &= 10 - \cos^2\left(\frac{x}{3} - \frac{\pi}{2}\right) - 2\cos\left(\frac{x}{3} - \frac{\pi}{2}\right) - 2 \\ &= 8 - 2\sin\left(\frac{x}{3}\right) - \sin^2\left(\frac{x}{3}\right), \end{aligned}$$

where in the last line we used complementary angle identity $\cos(\theta - \frac{\pi}{2}) = \sin(\theta)$.

Question 3e (3 marks)

MARK 1. Expresses vertical distance in terms of f and g, or equivalent merit
MARK 2. Provides correct answer for x-coordinates where vertical distance is maximum
MARK 3. Provides correct answer for maximum vertical distance

The vertical distance between the graphs of f and g at x is given by

$$d(x) = g(x) - f(x) = 5 - 2\cos\left(\frac{x}{3}\right) - 2\sin\left(\frac{x}{3}\right).$$

Then, the critical points of d are given by

$$d'(x) = 0 \implies x = \frac{3\pi}{4}(4k - 3), \quad k \in \mathbb{Z}.$$

These critical points include where the distance is minimum. Using a graph for aid, the maximum vertical distance occurs where

$$x = \frac{3\pi}{4}(8k - 3), \quad k \in \mathbb{Z},$$

and this maximum vertical distance is

$$d_{\max} = d\left(\frac{15\pi}{4}\right) = 2\sqrt{2} + 5.$$

Question 4a.i (1 mark)

MARK 1. Provides correct answer

Let $B \sim N\left(67, 1.2^2\right)$ be the diameter of the tennis ball. Then,

$$\Pr(65.4 \le B \le 68.6) = 0.8176 \quad \text{(4DP)}.$$

Question 4a.ii (2 marks)

MARK 1. Applies conditional probability definition, or equivalent merit

MARK 2. Provides correct answer

Using **part a.i**, we have

$$\Pr(B \geq 68 \mid 65.4 \leq B \leq 68.6) = \frac{\Pr(68 \leq B \leq 68.6)}{\Pr(65.4 \leq B \leq 68.6)} = \frac{0.1111...}{0.8176...} = 0.1359 \quad \text{(4DP)}.$$

Question 4b (3 marks)

MARK 1. Determines tail probability, or equivalent merit

MARK 2. Utilises inverse normal cumulative distribution function

MARK 3. Provides correct answer

Let $T \sim N\left(67, \sigma^2\right)$ be the diameter of a tennis ball produced with the new equipment. Then, by symmetry

$$\Pr(65.4 \leq T \leq 68.6) = 0.95 \implies \Pr(T < 65.4) = \Pr\left(Z < \frac{65.4 - 67}{\sigma}\right) = 0.025.$$

Using the inverse normal cumulative distribution function,

$$\frac{65.4 - 67}{\sigma} = -1.959964... \implies \sigma = 0.82 \text{ mm}.$$

Question 4c (2 marks)

MARK 1. Relates $\hat{P}$ to number of tennis balls not meeting diameter requirement

MARK 2. Provides correct answer

Let $X \sim \text{Bi}(75, 0.05)$ be the random variable representing the number of tennis balls not meeting the diameter requirement. Then, $X \overset{\text{d}}{=} 75\hat{P}$ and

$$\Pr\left(\hat{P} > 0.1\right) = \Pr(X \geq 8) = 0.0336 \quad \text{(4DP)}.$$

Question 4d.i (2 marks)

MARK 1. Provides correct answer for expected value

MARK 2. Provides correct answer for standard deviation

From the formula sheet,

$$\text{E}\left(\hat{P}\right) = 0.05 = \frac{1}{20} \quad \text{and} \quad \text{sd}\left(\hat{P}\right) = \sqrt{\frac{0.05(1-0.05)}{75}} = \frac{\sqrt{57}}{300}.$$

Question 4d.ii (2 marks)

MARK 1. Expresses probability in terms of number of tennis balls not meeting diameter requirement

MARK 2. Provides correct answer

From **part d.i**, we have

$$\Pr\left(\frac{1}{20} - \frac{\sqrt{57}}{300} \leq \hat{P} \leq \frac{1}{20} + \frac{\sqrt{57}}{300}\right) = \Pr(2 \leq X \leq 5) = 0.7220 \quad \text{(4DP)}.$$

Question 4e (1 mark)

MARK 1. Provides correct answer

By trial and error, the mode of $\hat{P}$ is $\frac{1}{25}$.

Question 4f (1 mark)

MARK 1. Provides correct answer

A 95% confidence interval is

$$\begin{aligned}\text{CI} &= \left(\frac{235}{240} - 1.95996...\sqrt{\frac{\frac{235}{240}\left(1-\frac{235}{240}\right)}{240}},\ \frac{235}{240} + 1.95996...\sqrt{\frac{\frac{235}{240}\left(1-\frac{235}{240}\right)}{240}}\right) \\ &= (0.9611, 0.9972), \quad \text{(4DP)}.\end{aligned}$$

Question 4g (1 mark)

MARK 1. Provides correct answer with justification

Yes, since $0.95 \notin (0.9611, 0.9972)$, the confidence interval suggests that the proportions of interest are different.

Question 5a (1 mark)

MARK 1. Provides correct answer

The normal to the graph of f at P is given by

$$y = -\frac{1}{f'(a)}(x-a) + f(a) \implies y = \frac{a^2+2}{2} - \frac{x}{a}.$$

Question 5b (2 marks)

MARK 1. Finds x-coordinate of Q, or equivalent merit
MARK 2. Provides correct answer

We have that

$$\frac{a^2+2}{2} - \frac{x}{a} = f(x) \implies x = a,\ -\frac{a^2+2}{a}.$$

Of course, a is the x-coordinate of P, so

$$f\left(-\frac{a^2+2}{a}\right) = \frac{\left(a^2+2\right)^2}{2a^2} \implies Q = \left(-\frac{a^2+2}{a},\ \frac{\left(a^2+2\right)^2}{2a^2}\right).$$

Question 5c (3 marks)

MARK 1. Reasons valid method
MARK 2. Finds area of at least one relevant region, or equivalent merit
MARK 3. Provides correct answer

Let Y be the y-axis intercept of the normal to the graph of f at P. Then, the area of the triangle OPQ is equal to the sum of the areas of the triangles OPY and OQY. That is,

$$A(a) = \frac{1}{2} \cdot \frac{a^2+2}{2} \cdot a + \frac{1}{2} \cdot \frac{a^2+2}{2} \cdot \frac{a^2+2}{a} = \frac{a^4+3a^2+2}{2a}.$$

Question 5d (1 mark)

MARK 1. Provides correct answer

Noting that $a > 0$, the area of the triangle is minimised when

$$A'(a) = \frac{3a^4+3a^2-2}{2a^2} = 0 \implies a = \sqrt{\frac{\sqrt{33}-3}{6}}.$$

Question 5e (2 marks)

MARK 1. Reasons valid method
MARK 2. Provides correct answer

The angle QOP is $90°$ when the product of the gradients of the line segments OP and OQ is -1. That is, again noting that $a > 0$,

$$\frac{\frac{a^2}{2}-0}{a-0} \cdot \frac{\frac{\left(a^2+2\right)^2}{2a^2}-0}{-\frac{a^2+2}{a}-0} = -1 \implies a = \sqrt{2}.$$

Question 5f (2 marks)

MARK 1. Reasons valid method
MARK 2. Provides correct answer

This problem is the same optimisation problem but reflected about the line $y = x$. Therefore, by **part d**,

$$b = f\left(\sqrt{\frac{\sqrt{33}-3}{6}}\right) = \frac{\sqrt{33}-3}{12}.$$

End of Solutions Book

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